Exercise 9

Let $z = re^{i\theta}$ be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 9.)

Solution

Evaluate $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ separately, assuming that $z = re^{i\theta}$. Let Θ be the principal argument ($-\pi < \Theta \leq \pi$) of z, and let $k = 0, 1, \ldots, m-1$.

$$(z^{1/m})^{-1} = [(re^{i\theta})^{1/m}]^{-1} = [(re^{i(\Theta+2\pi k)})^{1/m}]^{-1} = \left[r^{1/m}\exp\left(i\frac{\Theta+2\pi k}{m}\right)\right]^{-1}$$

$$= r^{-1/m}\exp\left(-i\frac{\Theta+2\pi k}{m}\right)$$

Use the formula for z^{-1} in Exercise 7 of Sec. 9.

$$(z^{-1})^{1/m} = \left[\left(\frac{1}{r}\right)e^{i(-\theta)}\right]^{1/m} = [r^{-1}e^{i(-\Theta+2\pi k)}]^{1/m} = r^{-1/m}\exp\left(-i\frac{\Theta-2\pi k}{m}\right)$$

Whether $2\pi k$ is added to Θ or subtracted from Θ, the value of the exponential is the same. Consequently, $(z^{1/m})^{-1} = (z^{-1})^{1/m}$.